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Prepotentials in $N = 2$ Supersymmetric $SU(3)$ YM-Theory with Massless Hypermultiplets*

Holger Ewen[1], Kristin Förger[1], Stefan Theisen[1,2]

[1] *Sektion Physik, Universität München*
Theresienstr. 37, 80333 München, Germany

[2] *CERN, Theory Division, Geneva, Switzerland*

Abstract

We explicitly determine the instanton corrections to the prepotential for $N = 2$ supersymmetric $SU(3)$ Yang-Mills theory with massless hypermultiplets in the weak coupling regions $u \to \infty$ and $v \to \infty$. We construct the Picard-Fuchs equations for $N_f < 6$ and calculate the monodromies using Picard-Lefschetz theorem for $N_f = 2, 4$. For all $N_f < 6$ the instanton corrections to the prepotential are determined using the relation between $\mathrm{Tr}\langle\phi^2\rangle$ and the prepotential.

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1 Introduction

In the last two years duality has become a very important tool in supersymmetric gauge theories as well as in string theory. The basic idea was developed by Seiberg and Witten [1] (for reviews, see e.g. Bilal [2] or Lerche [3]) who found, at the two-derivative level, the exact non-perturbative low energy Wilsonian effective action of $N = 2$ supersymmetric gauge theory with gauge group $SU(2)$ by using duality and the selfconsistent assumption of massless monopoles and dyons in the strong coupling region of the moduli space $\mathcal{M}$.

The main technical point in [1] is that the moduli space $\mathcal{M}$ coincides with the moduli space of an auxiliary elliptic curve[1]. Subsequently, generalizations to gauge groups $SU(N_c)$ without [6, 7, 8] and with matter (in the fundamental representation) [9, 10, 11, 12] have been worked out. Extensions to other groups, $SO(N_c)$ and $Sp(N_c)$ [13, 14] as well as to exceptional groups [15, 16] are also known.

The field content of $N = 2$ gauge theories for arbitrary gauge group G consists of an $N = 2$ chiral multiplet which contains a vector field, two Weyl fermions and a complex scalar, all in the adjoint representation of the gauge group. In addition to the gauge sector we can have N_f hypermultiplets, each containing two Weyl fermions and two complex bosons, transforming in some representation of G. The theory has flat directions with nonvanishing expectation values for $\mathrm{Tr}\phi^2$, along which the gauge group is generically broken to the Cartan subalgebra, e.g. $SU(3)$ is broken to $U(1)^{\otimes 2}$.

The Wilsonian low energy effective action is specified by a single holomorphic prepotential $\mathcal{F}$ and can be expressed in terms of $N = 1$ chiral multiplets A_i, whose scalar component we denote by a_i, and $N = 1$ $U(1)$ gauge multiplets $W_{\alpha i}$ $(i = 1, \ldots, N_c - 1)$:

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\mathrm{Im}\Big\{ \int d^4\theta\, \partial_i\mathcal{F}(A)\, \bar{A}^i + \frac{1}{2}\int d^2\theta\, \partial_i\partial_j\mathcal{F}(A)\, W^i_\alpha W^{\alpha j}\Big\} \tag{1}$$

where $\partial_i\mathcal{F} = \dfrac{\partial\mathcal{F}}{\partial A^i}$, and

$$\partial_i\partial_j\mathcal{F}(a) = \tau_{ij}(a) = \left(\frac{8\pi i}{g^2} + \frac{\theta}{\pi}\right)_{ij} \tag{2}$$

is the field dependent coupling constant. The metric on the quantum moduli space $\mathcal{M}$, $ds^2 = \mathrm{Im}(da_{Di}d\bar{a}_i)$ where $a_{Di} = \frac{\partial\mathcal{F}}{\partial a_i}$ is the magnetic dual of a_i, has singularities at which the local effective action breaks down due to certain BPS states becoming massless. Loops in moduli space around these singularities yield monodromies of the section $\Pi = \left(\begin{smallmatrix}\vec{a}_D\\ \vec{a}\end{smallmatrix}\right)$. The global monodromy properties fix the bundle. Once we know $a_D(a)$, the prepotential can be obtained by integration. To obtain a and a_D we use the fact that they are period integrals of a particular meromorphic differential λ on a hyperelliptic curve whose period matrix is τ. In the weak coupling region the prepotential is generally given by $\mathcal{F} = \mathcal{F}_{\text{class}} + \mathcal{F}_{1-\text{loop}} + \mathcal{F}_{\text{inst}}$ [16]:

$$\mathcal{F}_{\text{class}} = \frac{\tau_{\text{class}}}{2}\sum_{\alpha\in\Delta_+}\langle\alpha, a\rangle^2$$

[1] The appearance of Riemann surfaces in the solution of $N = 2$ supersymmetric YM theory finds an 'explanation' in the string theory context [4, 5].

$$
\begin{aligned}
\mathcal{F}_{1-\text{loop}} &= \frac{i}{4\pi}\sum_{\alpha\in\Delta_+}\langle\alpha,a\rangle^2\ln\Big(\frac{\langle\alpha,a\rangle^2}{\Lambda_{N_f}^2}\Big) - N_f\frac{i}{8\pi}\sum_w\langle w,a\rangle^2\ln\Big(\frac{\langle w,a\rangle^2}{\Lambda_{N_f}^2}\Big) \\
\mathcal{F}_{\text{inst}} &= \sum_{n=1}^{\infty}\mathcal{F}_n(a)\Lambda_{N_f}^{(2N_c-N_f)n}
\end{aligned}
\tag{3}
$$

The sums are over the positive roots Δ_+ of G and the weights of the representation of the hypermultiplets, respectively.

The exact results obtained from duality predict the precise form of the instanton corrections to the effective action. In the weak coupling regime various checks of these results have been performed by explicit instanton calculations in the microscopic theory at the one and two instanton level [17, 18, 19, 20, 21, 22].

In this paper we compute the prepotential for $N=2$ supersymmetric $SU(3)$ gauge theory with $N_f<6$ flavors in the fundamental representation. Although both the perturbative and the nonperturbative instanton corrections can in principle be obtained by explicitly doing the period integrals, we are attacking this problem via the Picard-Fuchs equations for the periods of the appropriate hyperelliptic curves, following [7]. The period integrals then only have to be solved to leading orders to determine which linear combinations of the given system of solutions of the Picard-Fuchs equations correspond to (a_{Di},a_i). Given the periods in one patch of $\mathcal{M}$ we then get them everywhere in $\mathcal{M}$ by analytic continuation. It is however often easier to solve the Picard-Fuchs equations in various patches and again adjust coefficients by computing the period integrals to leading orders. The monodromy can then be read off from the periods and, as a check, shown to coincide up to conjugation with the monodromies obtained from the Picard-Lefschetz formula [7]. Using the methods outlined above, we explicitly compute the periods, the monodromies and the prepotential $\mathcal{F}$ for $N_f=2$ and $N_f=4$. The one and two instanton contributions will be given explicitly. Our analysis will, however, be restricted to the weak coupling regime. We then show, how the instanton corrections for the prepotential can alternatively be obtained from the Picard-Fuchs operators and the relation between $\mathrm{Tr}\langle\phi^2\rangle$ and $\mathcal{F}$, derived in [23, 24]. We do this explicitly for all $N_f<6$.

While this paper was being proofread, a paper [25] was posted on hep-th which treats the same problem by explicitly doing the period integrals.

2 Picard-Fuchs Operators for A_2 with $N_f<6$

We start from the hyperelliptic curves associated with the gauge group A_2 with matter in the form given in [26]. We restrict ourselves here to the case of massless matter where the curves for $N_f<6$ are given by:

$$
y^2 = W(x;u,v)^2 - F(x;\Lambda_{N_f}) \tag{4}
$$

Here $W(x;u,v)=x^3-ux-v$ is the A_2-type simple singularity with the identification

$$
\begin{aligned}
u &= \mathrm{Tr}\langle\phi^2\rangle \\
v &= \mathrm{Tr}\langle\phi^3\rangle
\end{aligned}
\tag{5}
$$

as the gauge invariant coordinates on the moduli space $\mathcal{M}$ [7, 6]. Under the anomaly free global subgroup $\mathbf{Z}_{4N_c-2N_f} \subset U(1)_R$ they have R charge 4 and 6 respectively. F is given by

$$F(x;\Lambda_{N_f}) = \Lambda_{N_f}^{6-N_f}(x - \delta_{N_f,5}\frac{\Lambda_5}{12})^{N_f} \tag{6}$$

Λ_{N_f} is the dynamically generated scale of the theory, which can be matched to the scale of the microscopic theory [22]. In the limit $\Lambda_{N_f} \to 0$ the curves corresponding to the classical moduli space are recovered.

The singularities of the quantum moduli space are the zero locus of the discriminant of the curve. The discriminants are:

$$\begin{aligned}
\Delta_{N_f=1} &= \Lambda_1^{15}\Big(-3125\ \Lambda_1^{15} + 256\ \Lambda_1^5\ u^5 + 22500\ \Lambda_1^{10}\ u\ v - 1024\ u^6\ v - 43200\ \Lambda_1^5 u^2 v^2 \\
&\quad +13824\ u^3\ v^3 - 46656\ v^5\Big) && (7)\\
\Delta_{N_f=2} &= \Lambda_2^{12}\ v^2\Big(-4\ \Lambda_2^6 + 12\ \Lambda_2^4\ u - 12\ \Lambda_2^2\ u^2 + 4\ u^3 - 27\ v^2\Big)\Big(-4\ \Lambda_2^6 - 12\ \Lambda_2^4\ u \\
&\quad -12\ \Lambda_2^2\ u^2 - 4\ u^3 + 27\ v^2\Big) && (8)\\
\Delta_{N_f=3} &= \Lambda_3^9\ v^3\Big(-108\ \Lambda_3^6\ u^3 - 1024\ u^6 - 729\ \Lambda_3^9\ v - 8640\ \Lambda_3^3\ u^3\ v - 8748\ \Lambda_3^6\ v^2 \\
&\quad +13824\ u^3\ v^2 - 34992\ \Lambda_3^3\ v^3 - 46656\ v^4\Big) && (9)\\
\Delta_{N_f=4} &= \Lambda_4^6\ v^4\Big(\Lambda_4^2\ u^2 + 4\ u^3 + 4\ \Lambda_4^3\ v + 18\ \Lambda_4\ u\ v - 27\ v^2\Big)\Big(-\Lambda_4^2\ u^2 - 4\ u^3 + 4\ \Lambda_4^3\ v \\
&\quad +18\ \Lambda_4\ u\ v + 27\ v^2\Big) && (10)\\
\Delta_{N_f=5} &= \tilde\Lambda_5^3\Big(-\tilde\Lambda_5^3 + \tilde\Lambda_5\ u + v\Big)^5\Big(-20601\tilde\Lambda_5^{12} + 67635\tilde\Lambda_5^{10}\ u - 78840\tilde\Lambda_5^8\ u^2 \\
&\quad +36288\ \tilde\Lambda_5^6\ u^3 - 5895\ \tilde\Lambda_5^4\ u^4 + 117\ \tilde\Lambda_5^2\ u^5 - 16\ u^6 + 62775\ \tilde\Lambda_5^9\ v - 143775\ \tilde\Lambda_5^7\ u\ v \\
&\quad +90045\ \tilde\Lambda_5^5\ u^2\ v - 17145\ \tilde\Lambda_5^3\ u^3\ v + 36\ \tilde\Lambda_5\ u^4\ v - 68121\ \tilde\Lambda_5^6\ v^2 + 71280\ \tilde\Lambda_5^4\ u\ v^2 \\
&\quad -18495\ \tilde\Lambda_5^2\ u^2\ v^2 + 216\ u^3\ v^2 + 12825\ \tilde\Lambda_5^3\ v^3 - 6075\ \tilde\Lambda_5\ u\ v^3 - 729\ v^4\Big) && (11)
\end{aligned}$$

where $\tilde\Lambda_5 = \frac{\Lambda_5}{12}$. Obviously the moduli space for odd number of flavors is much more complicated than the one for even flavors where the discriminant factorizes.

The meromorphic differential on the Riemann surface of genus 2 corresponding to the hyperelliptic curve (4) for $N_f < 6$ is [27]:

$$\begin{aligned}
\lambda &= \frac{x}{2\pi i y F}\Big(2FW' - WF'\Big) \\
&= \frac{x}{2\pi i y(x - \frac{\Lambda_5}{12}\delta_{N_f,5})}\Big\{(6-N_f)x^3 + (N_f-2)ux + N_f v - \delta_{N_f,5}\frac{\lambda_5}{6}(3x^2-u)\Big\} && (12)
\end{aligned}$$

The components of the section Π are given as integrals over the meromorphic differential

$$a_i = \int_{\alpha_i}\lambda \qquad \text{and} \qquad a_{Di} = \int_{\beta_i}\lambda \tag{13}$$

where α_i and β_i are a symplectic basis of homology 1-cycles on the curves (4), i.e. $\alpha_i \cap \beta_j = -\beta_j \cap \alpha_i = \delta_{ij}$ and $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0$.

The Picard-Fuchs operators constitute a system of partial differential operators of second order for the periods of a holomorphic differential. A basis of holomorphic differentials on the A_2-curves is $\partial_u\lambda$, $\partial_v\lambda$. We now proceed similar to [7] by considering first and second derivatives of $\partial_v\lambda$ with respect to u and v. This produces expressions of the form $\frac{\phi(x)}{y^n}$ where $\phi(x)$ are polynomials in x, u and v. The power of y in the denominator as well as the degree of the polynomials in the numerator can be reduced by various identities to one and four, respectively. Between these one can find two nontrivial linear combinations which vanish up to total derivatives. They give rise to the second order Picard-Fuchs equations $\tilde{\mathcal{L}}_i\partial_v\Pi = 0$, with $\tilde{\mathcal{L}}$ of the general form:

$$\begin{aligned}
\tilde{\mathcal{L}}_{(1)} &= c_1^{(1)}\partial_{uu}^2 + c_2^{(1)}\partial_{uv}^2 + c_3^{(1)}\partial_u + c_4^{(1)}\partial_v + c_5^{(1)} && (14)\\
\tilde{\mathcal{L}}_{(2)} &= c_1^{(2)}\partial_{vv}^2 + c_2^{(2)}\partial_{uv}^2 + c_3^{(2)}\partial_u + c_4^{(2)}\partial_v + c_5^{(2)} && (15)
\end{aligned}$$

where the coefficients $c_j^{(i)}$ are polynomials in u and v.

For the reduction procedure one uses an identity derived from the fact that the discriminant can be written in the form $\Delta = a(x)y^2 + b(x)(y^2)'$ where $a(x) = \sum_{i=0}^4 a_i x^i$ and $b(x) = \sum_{i=0}^5 b_i x^i$. The two identities (up to total derivatives) which were used are:
(i) the power of $1/y$ is reduced by two through

$$\frac{\phi(x)}{y^n} = \frac{1}{\Delta y^{n-2}}\Big\{a\phi + \frac{2}{n-2}(b\phi')\Big\} \tag{16}$$

(ii) the powers of x in the numerator can be reduced using

$$\frac{x^k}{y^l} = -\frac{x^{k-n-2}}{y^l(2-nl+2k)}\Big((2-l)x\varphi + 2(k-n+1)\psi\Big) \tag{17}$$

for $k \neq (nl-2)/2$, where φ and ψ are defined via $y^2 = x^n + \psi(x)$ and $(y^2)' = nx^{n-1} + \varphi(x)$ and $n = 6$ for the $SU(3)$ case.

Since we are treating massless matter, $(\vec{a}_D, \vec{a})$ transforms irreducibly under monodromy. It is therefore possible to find Picard-Fuchs operators $\mathcal{L}$ with $\partial_v\mathcal{L}_i\Pi = \tilde{\mathcal{L}}_i\partial_v\Pi = 0$. If we pull ∂_v through $\tilde{\mathcal{L}}_i$ we obtain the following set of Picard-Fuchs operators:[2]

$N_f = 1$:

$$\begin{aligned}
\mathcal{L}_{(1)} &= 16\Big(25\ \Lambda_1^5 u^2 - 84\ u^3 v - 405\ v^3\Big)\partial_{uu}^2 + \Big(-625\ \Lambda_1^{10} + 3300\ \Lambda_1^5\ u\ v\\
&\quad -3456\ u^2\ v^2\Big)\partial_{uv}^2 + 12v\Big(25\ \Lambda_1^5 - 36\ u\ v\Big)\partial_v + 4\Big(25\ \Lambda_1^5 - 84\ u\ v\Big) && (18)\\
\mathcal{L}_{(2)} &= 4\Big(25\ \Lambda_1^5\ u^2 - 84\ u^3\ v - 405\ v^3\Big)\partial_{vv}^2 + \Big(1125\ \Lambda_1^5\ v - 64\ u^4 - 2160\ u\ v^2\Big)\partial_{uv}^2\\
&\quad +4\Big(-16\ u^3 - 135\ v^2\Big)\partial_v - 180\ v && (19)
\end{aligned}$$

$N_f = 2$:

$$\mathcal{L}_{(1)} = \Big(-8\ \Lambda_2^4\ u + 8\ u^3 + 27\ v^2\Big)\partial_{uu}^2 + 6v\Big(\Lambda_2^4 + 3\ u^2\Big)\partial_{uv}^2 + 2\ u \tag{20}$$

[2]Picard-Fuchs operators for the curves given in [10] were derived in [11].

$$\mathcal{L}_{(2)} = 3v\Big(8\,u^3 + 27\,v^2 - 8\,\Lambda_2^4\,u\Big)\partial^2_{vv} + 4\Big(2(\Lambda_2^4 - u^2)^2 + 27\,u\,v^2\Big)\partial^2_{uv} + \Big(8\,u^3 + 27\,v^2 - 8\,\Lambda_2^4\,u\Big)\partial_v + 9\,v \quad (21)$$

$N_f = 3$:

$$\mathcal{L}_{(1)} = \Big(9\,\Lambda_3^6 + 72\,\Lambda_3^3\,v + 64\,u^3 + 144\,v^2\Big)\partial^2_{uu} + 4\,u^2\Big(-\Lambda_3^3 + 32\,v\Big)\partial^2_{uv} + 4u\Big(-\Lambda_3^3 - 4\,v\Big)\partial_v + 16\,u \quad (22)$$

$$\mathcal{L}_{(2)} = 9v\Big(9\,\Lambda_3^6 + 72\,\Lambda_3^3 v + 64\,u^3 + 144\,v^2\Big)\partial^2_v + u\Big(27\,\Lambda_3^6 + 540\,\Lambda_3^3 v + 256\,u^3 + 1728\,v^2\Big)\partial^2_{uv} + \Big(27\,\Lambda_3^6 + 216\,\Lambda_3^3 v + 256\,u^3 + 432\,v^2\Big)\partial_v + 36\Big(\Lambda_3^3 + 4\,v\Big) \quad (23)$$

$N_f = 4$:

$$\mathcal{L}_{(1)} = \Big(4\,\Lambda_4^4\,u + 31\Lambda_4^2\,u^2 + 60\,u^3 + 81\,v^2\Big)\partial^2_{uu} + 4v\Big(2\,\Lambda_4^4 + 15\,\Lambda_4^2\,u + 27u^2\Big)\partial^2_{uv} + 3v\Big(-2\,\Lambda_4^2 - 9\,u\Big)\partial_v + 4\,\Lambda_4^2 + 15\,u \quad (24)$$

$$\mathcal{L}_{(2)} = v\Big(4\,\Lambda_4^4 u + 31\,\Lambda_4^2 u^2 + 60\,u^3 + 81v^2\Big)\partial^2_{vv} + 2\Big(\Lambda_4^4 u^2 + 8\,\Lambda_4^2 u^3 + 9\,\Lambda_4^2 v^2 + 16\,u^4 + 54\,uv^2\Big)\partial^2_{uv} + \Big(2\,\Lambda_4^4 u + 16\,\Lambda_4^2 u^2 + 32\,u^3 + 27v^2\Big)\partial_v + 9v \quad (25)$$

$N_f = 5$:

$$\begin{aligned}\mathcal{L}_{(1)} = {} & \Big(1152\,\tilde\Lambda_5^9 + 1908\,\tilde\Lambda_5^7 u + 6228\,\tilde\Lambda_5^6 v - 1860\,\tilde\Lambda_5^5 u^2 - 4680\,\tilde\Lambda_5^4 uv + 492\,\tilde\Lambda_5^3 u^3 + \\ & 396\,\tilde\Lambda_5^3 v^2 + 1140\,\tilde\Lambda_5^2 u^2 v - 52\,\tilde\Lambda_5 u^4 - 306\,\tilde\Lambda_5\,uv^2 - 132\,u^3 v - 81\,v^3\Big)\partial^2_{uu} + \\ & \Big(-45\,\tilde\Lambda_5^{10} - 2070\,\tilde\Lambda_5^8 u - 3960\,\tilde\Lambda_5^7 v + 150\,\tilde\Lambda_5^6 u^2 - 1665\,\tilde\Lambda_5^5 uv + 354\,\tilde\Lambda_5^4 u^3 - 3960\,\tilde\Lambda_5^4 v^2 \\ & +1866\,\tilde\Lambda_5^3 u^2 v - 29\,\tilde\Lambda_5^2 u^4 + 2520\,\tilde\Lambda_5^2 uv^2 - 165\,\tilde\Lambda_5 u^3 v + 270\,\tilde\Lambda_5 v^3 - 216\,u^2 v^2\Big)\partial^2_{uv} + \\ & \Big(-630\,\tilde\Lambda_5^8 - 480\,\tilde\Lambda_5^6 u - 1755\,\tilde\Lambda_5^5 v + 306\,\tilde\Lambda_5^4 u^2 + 714\,\tilde\Lambda_5^3 uv - 16\,\tilde\Lambda_5^2 u^3 - 180\,\tilde\Lambda_5^2 v^2 - \\ & 15\,\tilde\Lambda_5 u^2 v + 81\,uv^2\Big)\partial_v + 15\,\tilde\Lambda_5^5 + 48\,\tilde\Lambda_5^3 u + 120\,\tilde\Lambda_5^2 v - 13\,\tilde\Lambda_5 u^2 - 33\,uv \qquad (26)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{(2)} = {} & \Big(1152\,\tilde\Lambda_5^9 + 1908\,\tilde\Lambda_5^7 u + 6228\,\tilde\Lambda_5^6 v - 1860\,\tilde\Lambda_5^5 u^2 - 4680\,\tilde\Lambda_5^4 uv + 492\,\tilde\Lambda_5^3 u^3 + 396\,\tilde\Lambda_5^3 v^2 \\ & +1140\,\tilde\Lambda_5^2 u^2 v - 52\,\tilde\Lambda_5 u^4 - 306\,\tilde\Lambda_5\,uv^2 - 132\,u^3 v - 81\,v^3\Big)\partial^2_{vv} + \Big(-4770\,\tilde\Lambda_5^8 + \\ & 4860\,\tilde\Lambda_5^6 u - 3285\,\tilde\Lambda_5^5 v - 2298\,\tilde\Lambda_5^4 u^2 + 2538\,\tilde\Lambda_5^3 uv + 648\,\tilde\Lambda_5^2 u^3 + 360\,\tilde\Lambda_5^2 v^2 - \\ & 465\,\tilde\Lambda_5 u^2 v - 80\,u^4 - 108\,uv^2\Big)\partial^2_{uv} + \Big(2340\,\tilde\Lambda_5^6 - 2112\,\tilde\Lambda_5^4 u + 252\,\tilde\Lambda_5^3 v + \\ & 672\,\tilde\Lambda_5^2 u^2 - 120\,\tilde\Lambda_5 uv - 80\,u^3 - 27\,v^2\Big)\partial_v + 3\Big(18\,\tilde\Lambda_5^3 - 8\,\tilde\Lambda_5 u - 3\,v\Big) \qquad (27)\end{aligned}$$

For the pure YM case $N_f = 0$ the Picard-Fuchs equations were determined in [7] where it was found that they form an Appell system of type F_4. For $N_f \neq 0$ we could not identify the system of Picard-Fuchs operators with any of the generalized hypergeometric systems discussed in the mathematical literature.

3 $N_f = 4$

3.1 Monodromies

The singularities of the moduli space are at the zero loci of the discriminant $\Delta_{N_f} = 0$. Here two or more roots of the curve $y^2 = \prod_{i=1}^{6}(x - e_i)$ coincide and the associated genus two Riemann surface becomes singular. That is, some homology cycle $\nu = \vec{q}\cdot\vec{\alpha} + \vec{g}\cdot\vec{\beta}$ with magnetic charge vector $\vec{g} = (g_1, g_2)^T$ and electric charge vector $\vec{q} = (q_1, q_2)^T$ vanishes, indicating that dyons with charge vector $\vec{\nu} = (\vec{g}, \vec{q})$ become massless. Monodromies around these singularities have this charge vector as their left eigenvector with eigenvalue one: $\vec{\nu} M_\nu = \vec{\nu}$. After the choice of a fixed base point in the moduli space, the monodromies from loops around various branches $\Delta_{N_f} = 0$ can be calculated from the Picard-Lefshetz formula [7], which says that the monodromy action for a given cycle γ is determined by the vanishing cycle ν of the singularity as $M_\nu : \gamma \to \gamma - (\gamma \cap \nu)\nu$. This can easily be calculated after decomposing the vanishing cycles into the homology basis α_i, β_i. This gives the monodromy in matrix form:

$$M_{(\vec{g},\vec{q})} = \begin{pmatrix} \mathbf{1} + \vec{q}\otimes\vec{g} & \vec{q}\otimes\vec{q} \\ \vec{g}\otimes\vec{g} & \mathbf{1} - \vec{g}\otimes\vec{q} \end{pmatrix} \tag{28}$$

We fix the homology basis as in figure 1:

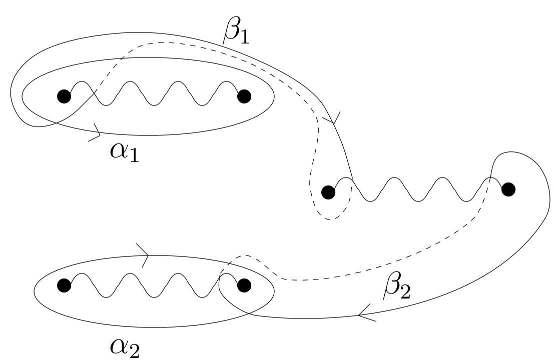

Figure 1: Basic cycles α_i and β_i in the x plane

In the $v = 1$ plane in the moduli space $\mathcal{M}_{N_f=4}$ we fix the reference point $u = -2$ and move to the six singular branches (see below), tracing the motion of the roots in the x plane. This results in the vanishing cycles shown in figure 2. They are

$$\begin{aligned} \nu_1 &= (1, 0, 0, 0) \\ \nu_2 &= (0, 1, -1, 2) \\ \nu_3 &= (1, 0, 2, -1) \\ \nu_4 &= (0, 1, 0, 0) \\ \nu_5 &= (1, 1, 1, 0) \\ \nu_6 &= (1, 1, 0, -1) \end{aligned} \tag{29}$$

from which one can easily calculate the monodromy using (28).

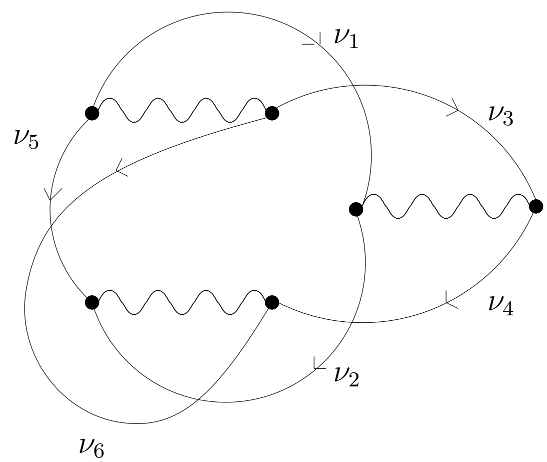

Figure 2: Vanishing cycles for $N_f = 4$

In figure 3 we give the structure of the moduli space $\mathcal{M}_{N_f=4}$. The figure shows the zeroes of the discriminant (10) with $\Lambda_4 = 1$ and real v, projected onto the plane Im$u = 0$. There appear cusps at the points $(u,v) = (0,0)$ and $(-\frac{1}{3}, \pm\frac{1}{27})$ and nodes at $(u,v) = (-\frac{1}{4}, 0)$ and $(-\frac{2}{9}, \pm\frac{2}{81\sqrt{3}})$. The four branches extending to the right of the vertical axis are real. The shown branches which extend to Re $u \to -\infty$ in fact each represent two branches with opposite Im $u \neq 0$. At the nodes the vanishing cycles do not intersect, that is $\nu_i \cap \nu_j = 0$ and the corresponding monodromies commute $[M_{\nu_i}, M_{\nu_j}] = 0$, in accordance with the van Kampen relations. This is a necessary condition for two dyons to condense simultaneously. They are then mutually local and can be described by a local effective action.

We look at two weak-coupling regions of the moduli space: one has large u for $v =$ const., the other has large v with $u =$ const.. Making a loop in the first asymptotic region, we encircle for generic v six lines, in the second case at $u > 0$ four lines. The monodromies for these regions correspond to going around all singular loci in the chosen plane, starting from the fixed base point. The semiclassical monodromy in the regions $u \to \infty$ and $v \to \infty$ are given by:

$$M_{u\to\infty} = r_2 \cdot r_3 \cdot r_1 = \begin{pmatrix} 0 & -1 & -5 & 6 \\ -1 & 0 & 8 & -5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{30}$$

$$M_{v\to\infty} = (r_3 \cdot M'_4 \cdot M'_3)^{-1} = \begin{pmatrix} -1 & -1 & -3 & 0 \\ 1 & 0 & 6 & -3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{31}$$

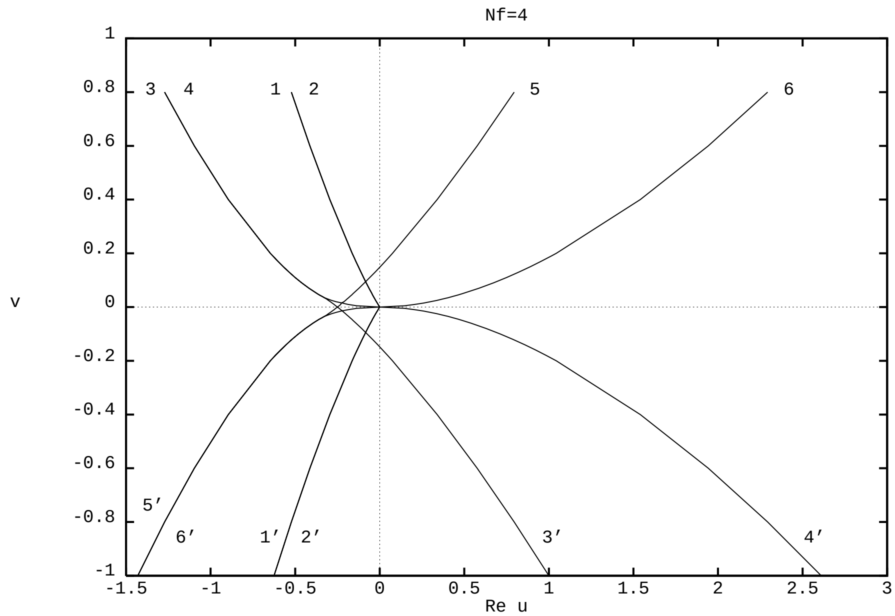

Figure 3: Moduli space for $N_f = 4$

where r_1, r_2, r_3 are the classical Weyl group generators:

$$r_1 \quad = \quad M_3 M_1 = \begin{pmatrix} -1 & 0 & 4 & -2 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

$$r_2 \quad = \quad M_2 M_4 = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{33}$$

$$r_3 \quad = \quad M_5 M_6 = \begin{pmatrix} 0 & -1 & 1 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{34}$$

and M_3' and M_4' are the monodromies associated to the vanishing cycles $\nu_3' = (0, 1, 0, -2)$ and $\nu_4' = (0, 1, -1, 0)$, where M_i' denote monodromies associated with branches with Re $v < 0$.

3.2 Solution in the semiclassical regions

We are now going to compute the period integrals in the two semiclassical regions $v \to \infty$ and $u \to \infty$. In each of these regions we find a basis for the solutions of the Picard-Fuchs equations consisting of two power series and two logarithmic solutions. To match them with the four periods we analytically compute the period integrals to leading orders.

For $v \to \infty$ we make the power series ansatz:

$$\omega = \sum_{n\geq 0, m\leq 0} c(n,m) u^{n+s} v^{m+t} \tag{35}$$

One finds $(s,t) = (0, \pm\frac{1}{3})$. For each set of indices one can find one power series solution and one logarithmic solution of the form:

$$\Omega = \omega \ln \frac{u^a}{v^b} + \sum_{n\geq 0, m\leq 0} d(n,m) u^{n+s} v^{m+t} \tag{36}$$

a and b are constants to be determined. As we did not find a solution in closed form of the recursion relations for $c(n,m)$ and $d(n,m)$, we give here only the first few terms of the solutions.

For $(s,t) = (0, \frac{1}{3})$ we get the following expansion:

$$\omega_1 = v^{1/3} - \frac{35}{104976}\frac{\Lambda_4^6}{v^{5/3}} - \frac{7}{1944}\frac{u}{v^{5/3}}\Lambda_4^4 - \frac{19019}{11019960576}\frac{\Lambda_4^{12}}{v^{11/3}} - \frac{\Lambda_4^2}{81}\frac{u^2}{v^{5/3}} - \frac{1}{81}\frac{u^3}{v^{5/3}} + \ldots \tag{37}$$

$$\begin{aligned}\Omega_1 &= \omega_1 \ln\frac{\Lambda_4^3}{v} - \frac{137}{209952}\frac{1}{v^{5/3}}\Lambda_4^6 - \frac{11}{1296}\frac{u}{v^{5/3}}\Lambda_4^4 - \frac{39737}{9183300480}\frac{1}{v^{11/3}}\Lambda_4^{12} \\ &\quad - \frac{\Lambda_4^2}{27}\frac{u^2}{v^{5/3}} - \frac{47279}{510183360}\frac{u}{v^{11/3}}\Lambda_4^{10} - \frac{1}{18}\frac{u^3}{v^{5/3}} + \ldots\end{aligned} \tag{38}$$

and for the second set of indices $(s,t) = (0, -\frac{1}{3})$:

$$\begin{aligned}\omega_2 &= \frac{\Lambda_4^2}{v^{1/3}} + 6\frac{u}{v^{1/3}} + \frac{385}{419904}\frac{\Lambda_4^8}{v^{7/3}} + \frac{55}{4374}\frac{u}{v^{7/3}}\Lambda_4^6 + \frac{279565}{449079842304}\frac{\Lambda_4^{14}}{v^{13/3}} \\ &\quad + \frac{5}{81}\frac{u^2}{v^{7/3}}\Lambda_4^4 + \frac{10}{81}\frac{u^3}{v^{7/3}}\Lambda_4^2 + \ldots\end{aligned} \tag{39}$$

$$\begin{aligned}\Omega_2 &= \omega_2 \ln\frac{\Lambda_4^3}{v} + \frac{5273}{1679616}\frac{\Lambda_4^8}{v^{7/3}} + \frac{103}{2187}\frac{u}{v^{7/3}}\Lambda_4^6 + \frac{9833323}{411411861504}\frac{\Lambda_4^{14}}{v^{13/3}} \\ &\quad + \frac{7}{27}\frac{u^2}{v^{7/3}}\Lambda_4^4 + \frac{11}{18}\frac{u^3}{v^{7/3}}\Lambda_4^2 + \ldots\end{aligned} \tag{40}$$

Having found the power series and the logarithmic solutions it remains to calculate the period integrals to leading orders, to determine the coefficients in $a_i = p_{i1}\,\omega_1 + p_{i2}\,\omega_2$ and $a_{Di} = q_{i1}\,\omega_1 + q_{i2}\,\omega_2 + q_{i3}\,\Omega_1 + q_{i4}\,\Omega_2$.

The first step in computing the period integrals $a_i = \int_{\alpha_i}\lambda$ and $a_{Di} = \int_{\beta_i}\lambda$ is to expand the six roots e_i of the curve $y^2 = \prod_i (x - e_i)$ around $v \to \infty$ with the result:

$$\begin{array}{ll} e_1 = \frac{\Lambda_4}{3} + v^{1/3} + \frac{1}{9v^{1/3}}\Lambda_4^2 + \ldots & e_4 = -\frac{\Lambda_4}{3} + v^{1/3} + \frac{1}{9v^{1/3}}\Lambda_4^2 + \ldots \\ e_2 = \frac{\Lambda_4}{3} + \zeta v^{1/3} + \frac{\zeta^2}{9v^{1/3}}\Lambda_4^2 + \ldots & e_5 = -\frac{\Lambda_4}{3} + \zeta v^{1/3} + \frac{\zeta^2}{9v^{1/3}}\Lambda_4^2 + \ldots \\ e_3 = \frac{\Lambda_4}{3} + \zeta^2 v^{1/3} + \frac{\zeta}{9v^{1/3}}\Lambda_4^2 + \ldots & e_6 = -\frac{\Lambda_4}{3} + \zeta^2 v^{1/3} + \frac{\zeta}{9v^{1/3}}\Lambda_4^2 + \ldots \end{array} \tag{41}$$

where $\zeta = e^{2\pi i/3}$.

The pairs of roots which correspond to the basic cycles chosen in figure 1 are:

$$\begin{array}{ll} \alpha_1 \rightarrow (e_5, e_2) & \qquad \beta_1 \rightarrow (e_5, e_4) \\ \alpha_2 \rightarrow (e_6, e_3) & \qquad \beta_2 \rightarrow (e_1, e_5) \end{array} \tag{42}$$

In order to treat the poles of λ in the integrals separately we split the integration region by introducing an arbitrary parameter ξ. At the end of the calculation the period integrals will have to be independent of ξ. Inserting the meromorphic one form (12), the integrals are of the form:

$$I_i = \int_{e_i}^{\xi} dx \frac{(x^3 + ux + 2v)}{\{\prod_{j=1}^{6}(x - e_j)\}^{1/2}} \tag{43}$$

If we now introduce $\Delta_1^{\pm} = \frac{1}{2}(e_1 \pm e_4)$, $\Delta_2^{\pm} = \frac{1}{2}(e_2 \pm e_5)$, $\Delta_3^{\pm} = \frac{1}{2}(e_3 \pm e_6)$, $\Delta_{i+3}^{\pm} = \pm\Delta_i^{\pm}$, change variables such that $x = \rho\Delta_i^- + \Delta_i^+$ and use the expansion $(x^3 + ux + 2v) = \sum_{k=0}^{3} \tilde{\epsilon}_k \rho^k$, we get:

$$I_i = \int_1^{\nu_i} d\rho \frac{\sum_{k=0}^{3} \tilde{\epsilon}_k \rho^k}{\sqrt{\rho^2 - 1}} \prod_{\substack{j \neq i \\ j \neq i-3}} \left\{ \frac{1}{(\Delta_j^+ - \Delta_i^+)(1 - \rho\epsilon_j)} \Big(1 + \frac{\sigma_j^2}{2(1 - \epsilon_j \rho)^2} + \ldots\Big) \right\} \tag{44}$$

where $\sigma_j = \frac{\Delta_j^-}{\Delta_j^+ - \Delta_i^+}$ and $\epsilon_j = \frac{\Delta_i^-}{\Delta_j^+ - \Delta_i^+}$ and $\nu_i = \frac{\xi - \Delta_i^+}{\Delta_i^-}$. We can now express I_i in terms of basic integrals $I_n^m = \int_1^{\nu} d\rho \frac{\rho^m}{\sqrt{\rho^2 - 1}(1 - \epsilon\rho)^n}$. By carefully doing the integrals in the complex plane, we get the following result for the periods:

$$a_1 = 2\int_{e_2}^{e_5} \lambda = 2\Big(\zeta\omega_1 + \frac{1}{18}\zeta^2\omega_2\Big) \tag{45}$$

$$a_2 = 2\int_{e_6}^{e_3} \lambda = 2\Big(\zeta^2\omega_1 + \frac{1}{18}\zeta\omega_2\Big) \tag{46}$$

$$\begin{aligned} a_{D1} = 2\int_{e_5}^{e_4} \lambda = {} & \frac{1}{i\pi 54}\Big\{\omega_2\Big(- 3\ln(108)(2 + \zeta) + i\pi(17\zeta + 8)\Big) \\ & +\omega_1\Big(- 54\ln(108)(1 - \zeta) - i\pi 18(17\zeta + 9) + 108(1 - \zeta)\Big) + 2(2 + \zeta)\Omega_2 \\ & +36(1 - \zeta)\Omega_1\Big\} \end{aligned} \tag{47}$$

$$\begin{aligned} a_{D2} = 2\int_{e_1}^{e_3} \lambda = {} & \frac{1}{i\pi 54}\Big\{\omega_2\Big(3\ln(108)(\zeta - 1) + i\pi(-13\zeta - 9)\Big) \\ & +\omega_1\Big(54\ln(108)(-\zeta - 2) + 18(4i\pi + 13i\pi\zeta + 6\zeta + 12)\Big) + 2(1 - \zeta)\Omega_2 \\ & +36(\zeta + 2)\Omega_1\Big\} \end{aligned} \tag{48}$$

Having expressed the periods in terms of power series and logarithmic solutions, we can check the monodromies by taking $v \rightarrow e^{2\pi i}v$, resulting in

$$\tilde{M}_{v\rightarrow\infty} = \begin{pmatrix} -1 & 1 & -5 & 0 \\ -1 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \tag{49}$$

This monodromy is $Sp(4,\mathbf{Z})$-conjugate to (31).

Similarly, we now construct solutions for the regime $u \to \infty$. There is one power series and one logarithmic solution for the indices $(s,t) = (1/2, 0)$:

$$\omega_1 = \sqrt{u} + \frac{1}{16}\frac{\Lambda_4^2}{\sqrt{u}} - \frac{3}{1024}\frac{\Lambda_4^4}{u^{3/2}} + \frac{5}{16384}\frac{\Lambda_4^6}{u^{5/2}} - \frac{175}{4194304}\frac{\Lambda_4^8}{u^{7/2}} - \frac{3}{8}\frac{v^2}{u^{5/2}} + \dots \tag{50}$$

$$\Omega_1 = \omega_1 \ln\frac{\Lambda_4^2}{u} - \frac{1}{8}\frac{\Lambda_4^2}{\sqrt{u}} + \frac{1}{1024}\frac{\Lambda_4^4}{u^{3/2}} + \frac{1}{49152}\frac{\Lambda_4^6}{u^{5/2}} - \frac{265}{25165824}\frac{\Lambda_4^8}{u^{7/2}} - \frac{v^2}{u^{5/2}} + \dots \tag{51}$$

and another set of solutions for $(s,t) = (-1,1)$:

$$\omega_2 = \frac{v}{u} + \frac{v^3}{u^4} + \frac{v^3}{u^5}\Lambda_4^2 + 3\frac{v^5}{u^7} + \frac{21}{2}\frac{v^5}{u^8}\Lambda_4^2 + \frac{21}{4}\frac{v^5}{u^9}\Lambda_4^4 + 12\frac{v^7}{u^{10}} + 90\frac{v^7}{u^{11}}\Lambda_4^2 + \dots \tag{52}$$

$$\Omega_2 = \omega_2 \ln\Big(\frac{v^8\Lambda_4^6}{u^{15}}\Big) + \frac{1}{4}\frac{v}{u^2}\Lambda_4^2 - \frac{1}{32}\frac{v}{u^3}\Lambda_4^4 + \frac{1}{192}\frac{v}{u^4}\Lambda_4^6 - \frac{1}{1024}\frac{v}{u^5}\Lambda_4^8 + \frac{119}{6}\frac{v^3}{u^4} + \frac{1}{5120}\frac{v}{u^6}\Lambda_4^{10} + \dots \tag{53}$$

The roots of the curve expand in the region $u \to \infty$ as:

$$\begin{aligned} e_1 &= -\frac{v}{u} + \frac{v^2}{u^3}\Lambda_4 + \dots & e_4 &= -\frac{v}{u} - \frac{v^2}{u^3}\Lambda_4 + \dots \\ e_2 &= -\frac{\Lambda_4}{2} + \sqrt{u} + \dots & e_5 &= \frac{\Lambda_4}{2} + \sqrt{u} + \dots \\ e_3 &= -\frac{\Lambda_4}{2} - \sqrt{u} + \dots & e_6 &= \frac{\Lambda_4}{2} - \sqrt{u} + \dots \end{aligned} \tag{54}$$

The integrals for the periods are calculated in the same manner as before.

$$a_1 = 2\int_{e_5}^{e_2} \lambda = 2\Big(-\frac{1}{2}\omega_2 - \omega_1\Big) \tag{55}$$

$$a_2 = 2\int_{e_6}^{e_3} \lambda = 2\Big(\frac{1}{2}\omega_2 - \omega_1\Big) \tag{56}$$

$$a_{D1} = 2\int_{e_2}^{e_1} \lambda = \frac{2}{i\pi}\Big(\omega_2\Big(2 + \frac{3\ln 2}{2}\Big) + \omega_1(-1 + 3\ln 2) - \frac{1}{4}\Omega_2 - \frac{1}{2}\Omega_1\Big) \tag{57}$$

$$a_{D2} = 2\int_{e_4}^{e_6} \lambda = \frac{2}{i\pi}\Big(\omega_2\Big(-2 - \frac{3\ln 2}{2}\Big) + \omega_1(-1 + 3\ln 2) + \frac{1}{4}\Omega_2 - \frac{1}{2}\Omega_1\Big) \tag{58}$$

The resulting monodromy matrix of the periods in the regime $u \to \infty$ conjugate to $M_{u\to\infty}$ (30) is:

$$\tilde{M}_{u\to\infty} = \begin{pmatrix} 0 & -1 & -7 & 8 \\ -1 & 0 & 8 & -7 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{59}$$

3.3 Prepotential

In the previous section we found two power series and two logarithmic solutions of the Picard-Fuchs operators for the semiclassical regions $u \to \infty$ and $v \to \infty$ and determined the periods $a_i(u,v)$ and $a_{Di}(u,v)$. In this section we calculate the prepotential $\mathcal{F}(a_1,a_2)$ by integration.

Since the periods $a_i(u,v)$ are expressed in terms of the Casimirs u and v, the prepotential is readily obtained by integrating the following two equations:

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial u} &= a_{Di}(u,v)\frac{\partial a_i(u,v)}{\partial u} &(60)\\ \frac{\partial\mathcal{F}}{\partial v} &= a_{Di}(u,v)\frac{\partial a_i(u,v)}{\partial v} &(61)\end{aligned}$$

The integrability condition $\frac{\partial a_{Di}}{\partial v}\frac{\partial a_i}{\partial u}-\frac{\partial a_{Di}}{\partial u}\frac{\partial a_i}{\partial v}=0$ can be used as a check for the period integrals a_i and a_{Di}. Integrating (60,61) yields the prepotential $\mathcal{F}(u,v)=\mathcal{F}_{\text{class}}(u,v)+\mathcal{F}_{1-\text{loop}}(u,v)+\mathcal{F}_{\text{inst}}(u,v)$, which contains power series in u, v and Λ_4, as well as logarithmic terms. Our aim is to express the prepotential $\mathcal{F}$ in terms of the periods a_i. For that we introduce the central charges $Z_i=\langle\vec{\alpha}_i,\vec{a}\rangle$ with $\vec{\alpha}_i\in\Delta_+(A_2)$ and $\vec{a}=(a_1,a_2)^T$:

$$\begin{aligned}Z_1 &= 2a_1-a_2\\ Z_2 &= 2a_2-a_1\\ Z_3 &= Z_1+Z_2\end{aligned}\qquad(62)$$

In these variables the prepotential is a homogenous function of degree two. The classical prepotential $\mathcal{F}_{\text{class}}$ is proportional to $\sum_{i=1}^3 Z_i^2$ and the one loop part contains logarithms of Z_i multiplied with homogenous polynomials in Z_i of degree two. The proportionality constants can be found by matching the expressions as functions of u and v against $\mathcal{F}(u,v)$. In this way one obtains simultaneously $\mathcal{F}_{\text{class}}$ and $\mathcal{F}_{1-\text{loop}}$ as functions of a_1 and a_2. For $u\to\infty$ we get:

$$\begin{aligned}\mathcal{F}_{\text{class}} &= \frac{1}{4i\pi}\sum_{i=1}^{3}Z_i^2 &(63)\\ \mathcal{F}_{1-\text{loop}} &= -\frac{1}{4i\pi}\sum_{i=1}^{3}Z_i^2\ln\Big(\frac{Z_i}{\Lambda_4}\Big)^2+\frac{1}{2i\pi}\Big\{\Big(\frac{Z_1-Z_2}{3}\Big)^2\ln\Big(\frac{Z_1-Z_2}{3\Lambda_4}\Big)^2+\\ &\quad+\Big(\frac{Z_2+Z_3}{3}\Big)^2\ln\Big(\frac{Z_2+Z_3}{3\Lambda_4}\Big)^2+\Big(\frac{Z_1+Z_3}{3}\Big)\ln\Big(\frac{Z_1+Z_3}{3\Lambda_4}\Big)^2\Big\}\end{aligned}$$

For $v\to\infty$ we find the same result up to an overall minus sign. This observation holds for all results in this and the following section.

After subtracting the classical and the one loop part from the prepotential a power series in Λ_4^2 remains which gives the instanton contributions. The individual contributions can be summed up in terms of Z_i. Here we give the result for the one and two instanton corrections, cf. (3):

$$\begin{aligned}\mathcal{F}_1 &= \frac{i}{3\pi}\Big(1-\frac{u_0^3}{\Delta_0}\Big) &(64)\\ \mathcal{F}_2 &= -\frac{i}{\pi}\Big(\frac{1}{6}\frac{u_0^2}{\Delta_0}-\frac{7}{6}\frac{u_0^5}{\Delta_0^2}+\frac{5}{2}\frac{u_0^8}{\Delta_0^3}\Big) &(65)\end{aligned}$$

where $u_0=a_1^2+a_2^2-a_1a_2=\frac{1}{6}\sum_i Z_i^2$ and $\Delta_0=\prod_i Z_i^2$.

4 $N_f = 2$

4.1 Monodromies

Proceeding similarly to the case $N_f = 4$ we determine the monodromies for $N_f = 2$ by fixing $v = 1$, $u = -2$ as a base point in the moduli space $\mathcal{M}_{N_f=2}$ and looping around the zero loci of $\Delta_{N_f=2}$. Tracing the motion of the roots of the curve in the x-plane leads to the vanishing cycles, from which we determine the monodromies.

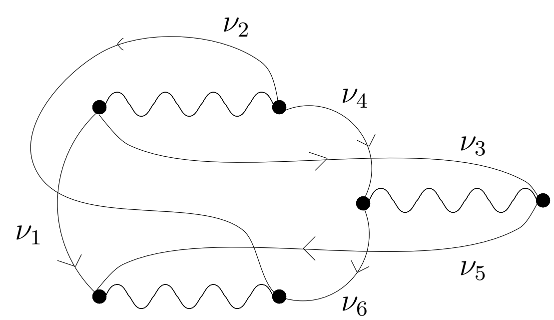

Figure 4: Vanishing cycles for $N_f = 2$

If we take the basic cycles to be the same as in the case $N_f = 4$, we get the following vanishing cycles:

$$\begin{aligned}
\nu_1 &= (1,1,1,0) \\
\nu_2 &= (1,1,2,1) \\
\nu_3 &= (1,0,3,-1) \\
\nu_4 &= (1,0,1,0) \\
\nu_5 &= (0,1,0,-1) \\
\nu_6 &= (0,1,-1,1)
\end{aligned} \tag{66}$$

In the case $N_f = 2$ the structure of the moduli space can easily be read off from (8), which reads for $\Lambda_2 = 1$

$$\Delta_{N_f=2} = v^2(4(u+1)^3 - 27v^2)(4(u-1)^3 - 27v^2) \tag{67}$$

There are two cusps at $u = \pm 1$, $v = 0$ at each of which 6 branches meet. In addition there are 4 nodes at the points $u = i/\sqrt{3}$, $v = \pm 9(1+i)/(8\sqrt[4]{3})$ and $u = -i/\sqrt{3}$, $v = \pm 9(1-i)/(8\sqrt[4]{3})$.

The semiclassical monodromy in the region $v \to \infty$ is:

$$M_{v\to\infty} = (M_1 \cdot M_7 \cdot M_6' \cdot M_5')^{-1} = \begin{pmatrix} -1 & -1 & -4 & 0 \\ 1 & 0 & 6 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{68}$$

where M_7 comes from $\nu_7 = (1, 1, 0, -1)$ and M_5' and M_6' coincide with M_5 and M_6. In the region $u \to \infty$ we get:

$$M_{u\to\infty} = M_6 \cdot M_5 \cdot M_2 \cdot M_1 \cdot M_3 \cdot M_4 = \begin{pmatrix} 0 & -1 & -7 & 8 \\ -1 & 0 & 8 & -7 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{69}$$

4.2 Solution in the semiclassical regions

The procedure is identical to the one in section 3.2, we will therefore be brief. Solving the recursion relation for $v \to \infty$ gives one power series and one logarithmic solution associated with the indices $s = 0$ and $t = \frac{1}{3}$:

$$\omega_1 = v^{1/3} - \frac{1}{54}\frac{u}{v^{5/3}}\Lambda_2^4 - \frac{5}{26244}\frac{\Lambda_2^{12}}{v^{11/3}} - \frac{1}{81}\frac{u^3}{v^{5/3}} - \frac{5}{1458}\frac{u^2}{v^{11/3}}\Lambda_2^8 - \frac{10}{2187}\frac{u^4}{v^{11/3}}\Lambda_2^4 + \ldots \tag{70}$$

$$\Omega_1 = \omega_1 \ln\frac{\Lambda_2^3}{v} + \frac{1}{36}\frac{u}{v^{5/3}}\Lambda_2^4 + \frac{7}{26244}\frac{\Lambda_2^{12}}{v^{11/3}} + \frac{1}{324}\frac{u^2}{v^{11/3}}\Lambda_2^8 + \ldots \tag{71}$$

A second solution can be found for $s = 0$ and $t = -\frac{1}{3}$:

$$\begin{aligned}\omega_2 = {} & \frac{u}{v^{1/3}} + \frac{1}{216}\frac{\Lambda_2^8}{v^{7/3}} + \frac{1}{27}\frac{u^2}{v^{7/3}}\Lambda_2^4 + \frac{175}{104976}\frac{u}{v^{13/3}}\Lambda_2^{12} + \frac{91}{5668704}\frac{1}{v^{19/3}}\Lambda_2^{20} + \frac{1}{81}\frac{u^4}{v^{7/3}} \\ & + \frac{175}{17496}\frac{u^3}{v^{13/3}}\Lambda_2^{12} + \frac{35}{4374}\frac{u^5}{v^{13/3}}\Lambda_2^4 + \ldots \end{aligned} \tag{72}$$

$$\begin{aligned}\Omega_2 = {} & \omega_2 \ln\frac{\Lambda_2^3}{v} + \frac{1}{96}\frac{\Lambda_2^8}{v^{7/3}} + \frac{1}{9}\frac{u^2}{v^{7/3}}\Lambda_2^4 + \frac{1945}{419904}\frac{u}{v^{13/3}}\Lambda_2^{12} + \frac{1}{18}\frac{u^4}{v^{7/3}} + \\ & \frac{85}{2592}\frac{u^3}{v^{13/3}}\Lambda_2^8 + \ldots \end{aligned} \tag{73}$$

We expand the roots in the semiclassical region $v \to \infty$, that are needed for calculating the integrals:

$$\begin{aligned} e_1 &= v^{1/3} + \frac{\Lambda_2 + u}{3v^{1/3}} + \ldots & e_4 &= v^{1/3} - \frac{\Lambda_2 - u}{3v^{1/3}} + \ldots \\ e_2 &= \zeta v^{1/3} + \frac{\zeta^2(\Lambda_2 + u)}{3v^{1/3}} + \ldots & e_5 &= \zeta v^{1/3} - \frac{\zeta^2(\Lambda_2 - u)}{3v^{1/3}} + \ldots \\ e_3 &= \zeta^2 v^{1/3} + \frac{\zeta(\Lambda_2 + u)}{3v^{1/3}} + \ldots & e_6 &= \zeta^2 v^{1/3} - \frac{\zeta(\Lambda_2 - u)}{3v^{1/3}} + \ldots \end{aligned} \tag{74}$$

The period integrals are calculated in a way parallel to the case $N_f = 4$ with the result:

$$a_1 = 2\int_{e_5}^{e_2} \lambda = 2\Big(\omega_1 \zeta + \frac{\omega_2}{3}\zeta^2\Big) \tag{75}$$

$$a_2 = 2\int_{e_6}^{e_3} \lambda = 2\Big(-\omega_1 \zeta^2 - \frac{\omega_2}{3}\zeta\Big) \tag{76}$$

$$
\begin{aligned}
a_{D1} = 2\int_{e_2}^{e_4}\lambda &= \frac{1}{9i\pi}\Big\{\omega_1\Big(-57i\pi\zeta - 27i\pi + 36(1-\zeta) + 9\ln(108)(\zeta-1)\Big) + \\
&\omega_2\Big(-3\ln 108(\zeta+2) + 19i\pi\zeta + 10i\pi\Big) + 12(1-\zeta)\Omega_1 + \\
&(\zeta+2)4\Omega_2\Big\} \qquad (77)\\
a_{D2} = 2\int_{e_1}^{e_6}\lambda &= -\frac{\zeta}{9i\pi}\Big\{\omega_1\Big(\ln(108)9(1+2\zeta) - 6i\pi(\zeta^2+2\zeta) + 21i\pi - 36(1+2\zeta)\Big) + \\
&\omega_2\Big(3\ln 108(\zeta+2) + 2i\pi\zeta^2 + 11i\pi\zeta\Big) - 12\Omega_1(1+2\zeta) - \\
&4\Omega_2(2+\zeta)\Big\} \qquad (78)
\end{aligned}
$$

They undergo the semiclassical monodromy

$$
M_{v\to\infty} = \begin{pmatrix} -1 & -1 & -6 & 0 \\ 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad (79)
$$

which is conjugated to (68).

The region $u \to \infty$ gives rise to a set of solutions with indices $(s = \frac{1}{2}, t = 0)$ and $(s = -1, t = 1)$. For $s = \frac{1}{2}$ and $t = 0$ we get one power series and one logarithmic solution:

$$
\begin{aligned}
\omega_1 &= \sqrt{u} - \frac{1}{16}\frac{\Lambda_2^4}{u^{3/2}} - \frac{15}{1024}\frac{\Lambda_2^8}{u^{7/2}} - \frac{3}{8}\frac{v^2}{u^{5/2}} - \frac{105}{16384}\frac{\Lambda_2^{12}}{u^{11/2}} - \frac{105}{128}\frac{v^2}{u^{9/2}}\Lambda_2^4 - \frac{15015}{4194304}\frac{\Lambda_2^{16}}{u^{15/2}} + \qquad (80)\\
\Omega_1 &= \omega_1\ln\frac{\Lambda_2^2}{u} + \frac{1}{16}\frac{\Lambda_2^4}{u^{3/2}} + \frac{13}{2048}\frac{\Lambda_2^8}{u^{7/2}} - \frac{1}{4}\frac{v^2}{u^{5/2}} + \frac{163}{98304}\frac{\Lambda_2^{12}}{u^{11/2}} - \frac{37}{128}\frac{v^2}{u^{9/2}}\Lambda_2^4 + \ldots \qquad (81)
\end{aligned}
$$

and a second set of solutions for $s = -1$ and $t = 1$:

$$
\begin{aligned}
\omega_2 &= \frac{v}{u} + \frac{1}{2}\frac{v}{u^3}\Lambda_2^4 + \frac{3}{8}\frac{v}{u^5}\Lambda_2^8 + \frac{v^3}{u^4} + \frac{5}{16}\frac{v}{u^7}\Lambda_2^{12} + 5\frac{v^3}{u^6}\Lambda_2^4 + \frac{35}{128}\frac{v}{u^9}\Lambda_2^{16} + \ldots \qquad (82)\\
\Omega_2 &= \omega_2\ln\frac{v\Lambda_2^3}{u^3} + \frac{3}{2}\frac{v}{u^3}\Lambda_2^4 + \frac{3}{2}\frac{v}{u^5}\Lambda_2^8 + \frac{25}{6}\frac{v^3}{u^4} + \frac{23}{16}\frac{v}{u^7}\Lambda_2^{12} + \frac{241}{12}\frac{v^3}{u^6}\Lambda_2^4 + \ldots \qquad (83)
\end{aligned}
$$

For the integration we expand the roots of the curve in the limit of large u:

$$
\begin{aligned}
e_1 &= -\frac{v}{u} - 3\frac{v}{u^2}\Lambda_2^2 + \ldots & e_4 &= -\frac{v}{u} + 3\frac{v}{u^2}\Lambda_2^2 + \ldots \\
e_2 &= \sqrt{u} + \frac{1}{2}\Big(-\frac{\Lambda_2^2}{\sqrt{u}} + \frac{v}{u}\Big) + \ldots & e_5 &= \sqrt{u} + \frac{1}{2}\Big(\frac{\Lambda_2^2}{\sqrt{u}} + \frac{v}{u}\Big) + \ldots \qquad (84)\\
e_3 &= -\sqrt{u} + \frac{1}{2}\Big(\frac{\Lambda_2^2}{\sqrt{u}} + \frac{v}{u}\Big) + \ldots & e_6 &= -\sqrt{u} + \frac{1}{2}\Big(-\frac{\Lambda_2^2}{\sqrt{u}} + \frac{v}{u}\Big) + \ldots
\end{aligned}
$$

For the period integrals we find:

$$
a_1 = 2\int_{e_5}^{e_2}\lambda = 2\Big(-\omega_1 - \frac{1}{2}\omega_2\Big) \qquad (85)
$$

$$
\begin{aligned}
a_2 &= 2\int_{e_3}^{e_6}\lambda = 2\Big(-\omega_1+\frac{1}{2}\omega_2\Big) &(86)\\
a_{D1} &= 2\int_{e_2}^{e_4}\lambda = \frac{2}{i\pi}\Big(\omega_1(-2+3\ln 2)+\omega_2\Big(1+\frac{3}{2}\ln 2\Big)-\Omega_1-\Omega_2\Big) &(87)\\
a_{D2} &= 2\int_{e_1}^{e_3}\lambda = \frac{2}{i\pi}\Big(\omega_1(-2+3\ln 2)+\omega_2\Big(-1-\frac{3\ln 2}{2}\Big)-\Omega_1+\Omega_2\Big) &(88)
\end{aligned}
$$

which leads to the following monodromy matrix:

$$
\tilde{M}_{u\to\infty} = \begin{pmatrix} 0 & -1 & -7 & 9 \\ -1 & 0 & 5 & -3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{89}
$$

This is conjugated to (69).

4.3 Prepotential

As in the $N_f = 4$ case, the instanton corrections to the prepotential are obtained by subtracting $\mathcal{F}_{\text{class}}$ and $\mathcal{F}_{1-\text{loop}}$ from $\mathcal{F}(u,v)$. For both weak-coupling regions we find:

$$
\begin{aligned}
\mathcal{F}_{\text{class}} &= \frac{1}{2i\pi}\sum_{i=1}^{3} Z_i^2\Big(1+\frac{\ln 2}{3}\Big) &(90)\\
\mathcal{F}_{1-\text{loop}} &= -\frac{1}{4i\pi}\sum_{i=1}^{3} Z_i^2 \ln\left(\frac{Z_i}{\Lambda_2}\right)^2 + \frac{1}{4i\pi}\Big\{\Big(\frac{Z_1-Z_2}{3}\Big)^2\ln\left(\frac{Z_1-Z_2}{3\Lambda_2}\right)^2 + &(91)\\
&\quad \Big(\frac{Z_2+Z_3}{3}\Big)^2\ln\left(\frac{Z_2+Z_3}{3\Lambda_2}\right)^2 + \Big(\frac{Z_1+Z_3}{3}\Big)^2\ln\left(\frac{Z_1+Z_3}{3\Lambda_2}\right)^2\Big\}
\end{aligned}
$$

The coefficients of Λ_2^4 and Λ_2^8 in the instanton series sum up to:

$$
\begin{aligned}
\mathcal{F}_1 &= -\frac{4i}{\pi}\frac{u_0^2}{\Delta_0} &(92)\\
\mathcal{F}_2 &= -\frac{i}{\pi}\Big(\frac{8}{\Delta_0}-112\frac{u_0^3}{\Delta_0^2}+360\frac{u_0^6}{\Delta_0^3}\Big) &(93)
\end{aligned}
$$

with u_0 and Δ_0 as before.

5 Instanton Corrections by an Alternative Method

So far we have seen how to determine the prepotential from the periods a_i and a_{Di}. This requires knowledge of all four solutions of the Picard-Fuchs equations. However, our ansatz for the logarithmic solutions does not work for all N_f. We now present an alternative way to derive the prepotential which requires only knowledge of the power series solutions of the

Picard-Fuchs equations.[3] We will apply this method to $N_f = 1, 3$ and 5 in the semiclassical region $u \to \infty$ and check our previous results for $N_f = 2$ and 4.

Classically, i.e. for $\Lambda_{N_f} \to 0$, the Casimirs are given by:

$$\begin{aligned} u_0 &= \frac{1}{6}\sum_{i=1}^{3} Z_i^2 = a_1^2 + a_2^2 - a_1 a_2 \\ v_0 &= a_1 a_2 (a_1 - a_2) \end{aligned} \tag{94}$$

Inverting the above equations we get:

$$\begin{aligned} a_1 &= p_+ + p_- \\ a_2 &= -\zeta p_+ - \zeta^2 p_- \end{aligned} \tag{95}$$

where $p_\pm = \left(\frac{v_0}{2} \pm \frac{1}{2}\sqrt{v_0^2 - \frac{4}{27}u_0^3}\right)^{1/3}$. On the other hand, evaluating a_i for $u_0 \to \infty$ we get $a_1 \sim \sqrt{u_0} + \frac{1}{2}\frac{v_0}{u_0}$ and $a_2 \sim \sqrt{u_0} - \frac{1}{2}\frac{v_0}{u_0}$.

Since we know that in this region the Picard-Fuchs equations give two power series solutions ω_1 and ω_2 with asymptotic behaviour $\omega_1 = \sqrt{u} + \ldots$ and $\omega_2 = \frac{v}{u} + \ldots$, the periods must be of the form $a_1 = -2(\omega_1 + \frac{1}{2}\omega_2)$ and $a_2 = -2(\omega_1 - \frac{1}{2}\omega_2)$, taking the normalization from our previous conventions for the period integrals.

Now we use a relation between u and $\mathcal{F}$ derived in [23, 24, 28] to obtain the instanton corrections of the prepotential:

$$\begin{aligned} u(Z) &= \frac{4i\pi}{(2N_c - N_f)}\Big(\mathcal{F} - \frac{1}{2} Z_j \frac{\partial \mathcal{F}}{\partial Z_j}\Big) \\ &= \frac{1}{6}\sum_{i=1}^{3} Z_i^2 + 2i\pi \sum_{n=1}^{\infty} \mathcal{F}_n(Z) n \Lambda_{N_f}^{(2N_c - N_f)n} \end{aligned} \tag{96}$$

u itself has a power series expansion in Λ_{N_f}, namely $u(Z) = u_0 + \sum_{n=1}^{\infty} \mathcal{G}_n(Z) \Lambda_{N_f}^{(2N_c - N_f)n}$ with u_0 defined in (94). Summing the series for u by rewriting it in the variables Z_i yields the instanton corrections to the prepotential.

For $N_f = 1, \ldots, 5$ we determine the two power series ω_1 and ω_2 again by using the Picard-Fuchs equations and find for u

$$\begin{aligned}
N_f = 1: \quad u &= u_0 - 18\frac{v_0}{\Delta_0}\Lambda_1^5 - \Big(1008\frac{u_0^2}{\Delta_0^2} - 4320\frac{u_0^5}{\Delta_0^3}\Big)\Lambda_1^{10} + \mathcal{O}(\Lambda_1^{15}) \\
N_f = 2: \quad u &= u_0 + 8\frac{u_0^2}{\Delta_0}\Lambda_2^4 + 4\Big(\frac{8}{\Delta_0} - 112\frac{u_0^3}{\Delta_0^2} + 360\frac{u_0^6}{\Delta_0^3}\Big)\Lambda_2^8 + \mathcal{O}(\Lambda_2^{12}) \\
N_f = 3: \quad u &= u_0 - \frac{3}{2}\frac{u_0 v_0}{\Delta_0}\Lambda_3^3 - \Big(-\frac{2}{3}\frac{u_0}{\Delta_0} + \frac{35}{3}\frac{u_0^4}{\Delta_0^2} - 30\frac{u_0^7}{\Delta_0^3}\Big)\Lambda_3^6 + \mathcal{O}(\Lambda_3^9) \\
N_f = 4: \quad u &= u_0 - \frac{2}{3}\Big(1 - \frac{u_0^3}{\Delta_0}\Big)\Lambda_4^2 + 4\Big(\frac{1}{6}\frac{u_0^2}{\Delta_0} - \frac{7}{6}\frac{u_0^5}{\Delta_0^2} + \frac{5}{2}\frac{u_0^8}{\Delta_0^3}\Big)\Lambda_4^4 + \mathcal{O}(\Lambda_4^6) \\
N_f = 5: \quad u &= u_0 - \frac{3}{2}\frac{u_0^2 v_0}{\Delta_0}\Lambda_5 - \Big(-\frac{10}{3} - 2\frac{u_0^3}{\Delta_0} + \frac{49}{3}\frac{u_0^6}{\Delta_0^2} - 30\frac{u_0^9}{\Delta_0^3}\Big)\Lambda_5^2 + \mathcal{O}(\Lambda_5^3)
\end{aligned} \tag{97}$$

[3] This method can also be applied to other patches of the moduli space.

From these equations and (96) we can read off the one and two instanton corrections for all N_f. For the two cases $N_f = 2, 4$ calculated at length above we can compare the results. For example for $N_f = 2$ we have $\mathcal{F}_1 = \frac{\mathcal{G}_1}{2i\pi} = -\frac{4i}{\pi}\frac{u_0^2}{\Delta_0}$ and $\mathcal{F}_2 = \frac{\mathcal{G}_2}{4i\pi} = -\frac{i}{\pi}\Big(\frac{8}{\Delta_0} - 112\frac{u_0^3}{\Delta_0^2} + 360\frac{u_0^6}{\Delta_0^3}\Big)$ which agrees with (92). For $N_f = 4$ we find agreement with the results (64).

For the one and two instanton contributions for $N_f = 1, 3, 5$ we find:

$$\begin{aligned} N_f = 1: \qquad \mathcal{F}_1 &= 9\frac{i}{\pi}\frac{v_0}{\Delta_0} && (98) \\ \mathcal{F}_2 &= \frac{i}{\pi}\Big(252\frac{u_0^2}{\Delta_0^2} - 1080\frac{u_0^5}{\Delta_0^3}\Big) \\ N_f = 3: \qquad \mathcal{F}_1 &= \frac{3}{4}\frac{i}{\pi}\frac{u_0 v_0}{\Delta_0} && (99) \\ \mathcal{F}_2 &= \frac{i}{\pi}\Big(-\frac{1}{6}\frac{u_0}{\Delta_0} + \frac{35}{12}\frac{u_0^4}{\Delta_0^2} - \frac{15}{2}\frac{u_0^7}{\Delta_0^3}\Big) \\ N_f = 5: \qquad \mathcal{F}_1 &= \frac{3}{4}\frac{i}{\pi}\frac{u_0^2 v_0}{\Delta_0} && (100) \\ \mathcal{F}_2 &= \frac{i}{\pi}\Big(-\frac{5}{6} - \frac{u_0^3}{2\Delta_0} + \frac{49}{12}\frac{u_0^6}{\Delta_0^2} - \frac{15}{2}\frac{u_0^9}{\Delta_0^3}\Big) && (101) \end{aligned}$$

With both methods we can easily calculate higher order instanton corrections as well. It would be nice to verify these results by explicit instanton calculation.

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